Functions – Rational Functions

We've looked at functions like: $\frac{x^2-2x+5}{x+1}$, which are made by dividing a polynomial by another one.

These are called rational functions. A rational function is a function of the form $\frac{p(x)}{q(x)}$,

where p(x) and q(x) are polynomials.

The most basic rational function is the reciprocal function: $f(x) = \frac{1}{x}$

Graphing rational functions:

Ex: $f(x) = \frac{6}{x-3}$

Start by graphing some points:

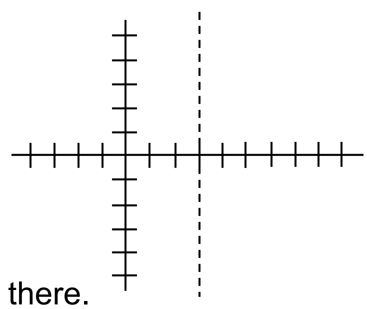
$x = -3,\ f(-3) = \frac{6}{-3-3} = \frac{6}{-6} = -1$

$x = 1,\ f(1) = \frac{6}{1-3} = \frac{6}{-2} = -3$

$x = 3,\ f(3) = \frac{6}{3-3} = \frac{6}{0} = Undefined$

$x = 5,\ f(5) = \frac{6}{5-3} = \frac{6}{2} = 3$

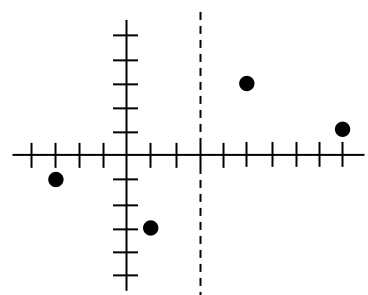
$x = 9,\ f(1) = \frac{6}{9-3} = \frac{6}{6} = 1$



Note: This function DOES NOT EXIST everywhere.

The graph can NOT go through x=3.

We'll draw a line at x = 3 to remind ourselves that the graph can't cross there.

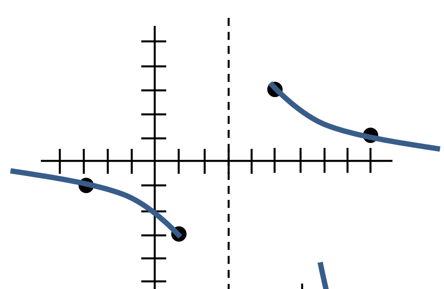


Plotting the other points, we see that there is something happening to the left of the line, and there is something happening to the right of the line.

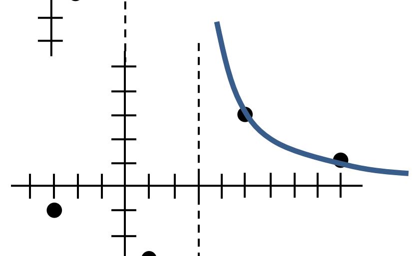
Next, look at the end behavior.

$x = -\infty,\ f(-\infty) = \frac{6}{-\infty-3} = \frac{6}{-\infty} \approx 0$

$x = \infty,\ f(\infty) = \frac{6}{\infty-3} = \frac{6}{\infty} \approx 0$



Lastly, what's going on as the curve approaches



the dotted line (from the left and right).

$x = 2.99,\ f(2.99) = \frac{6}{2.99-3} = \frac{6}{-0.01} = -600$

$x = 3.01, \qquad f(3.01) = \frac{6}{3.01-3} = \frac{6}{0.01} = 600$

As the graph approaches x = 3,

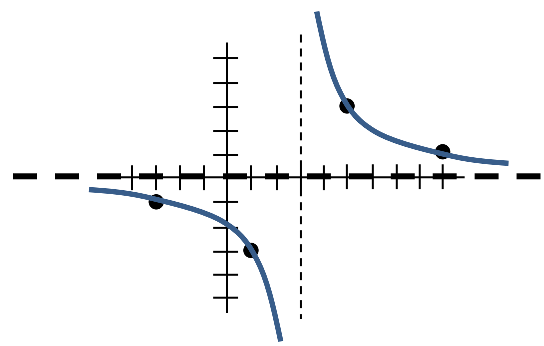
the function approaches $-\infty$ on the left and ∞ on the right.

The graph gets CLOSE to x = 3, but never actually crosses the dotted line.

We say that the function has an asymptote at x = 3.

More specifically, this is a vertical asymptote because the dotted line is vertical.

If you notice, the value of the function approaches zero on the ends, but never is actually zero.
There is an asymptote at y = 0.
This is a horizontal asymptote.



Most rational functions have asymptotes.

A rational function has a VERTICAL ASYMPTOTE wherever the function is undefined.

This occurs wherever the denominator is zero.

A rational function, $f(x) = \frac{p(x)}{q(x)} = \frac{ax^m+\cdots}{bx^n+\cdots}$, can be made up of:

a polynomial in the numerator of <u>degree m</u> with leading coefficient a, and

a polynomial in the denominator of <u>degree n</u> with leading coefficient b.

If $< n$, there is a HORIZONTAL ASYMPTOTE at $y = 0$

If $= n$, there is a HORIZONTAL ASYMPTOTE at $y = \frac{a}{b}$

If $m > n$, there is a NO horizontal asymptote.

BUT, if $m > n$ by ONLY 1, then there is what is called a SLANT ASYMPTOTE.

This is a line that the graph does not cross, that is neither vertical nor horizontal.

Ex: Graph $f(x) = \frac{x+5}{x^2+x-6}$

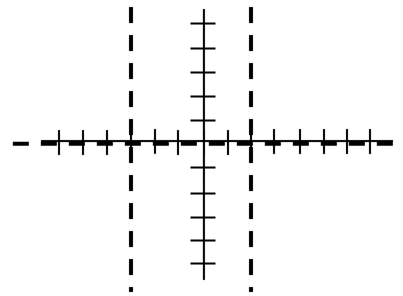
1. Find any vertical asymptotes. (Where is the denominator = 0 ?)

$x^2 + x - 6 = 0$ Factor $(x+3)(x-2) = 0$ $x = -3,\ x = 2$

Therefore, there are **two** vertical asymptotes:

one at $x = -3$,

and one at $x = 2$



2. Find any horizontal asymptotes.

Numerator: $x + 5$ has a degree of 1 = m.

Denominator: $x^2 + x - 6$ has a degree 2 = n.

Since $m < n$, there is a horizontal asymptote at y = 0.

3. Figure out what the different pieces of the graph look like.

Pick points that are less than x = –3:

$$f(-4) = \frac{-4+5}{(-4)^2 + (-4) - 6} = \frac{1}{6}$$

$$f(-6) = \frac{-6+5}{(-6)^2+(-6)-6} = \frac{-1}{24}$$

Pick points between –3 and 2:

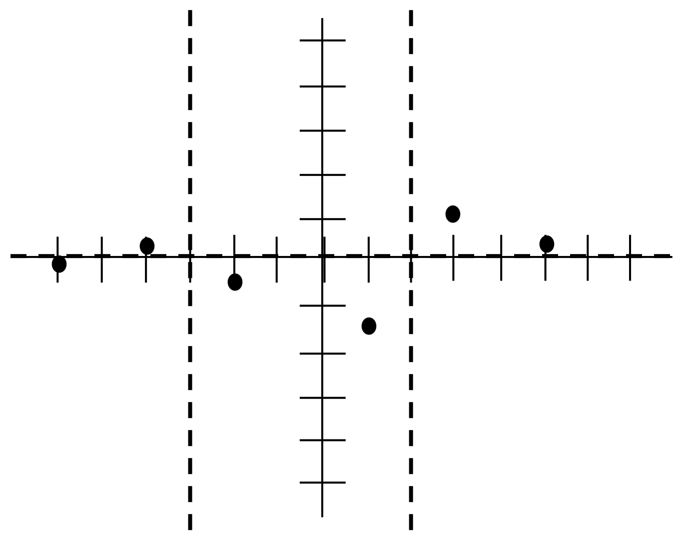
$$f(-2) = \frac{-2+5}{(-2)^2 + (-2) - 6} = \frac{3}{-4} = \frac{-3}{4}$$

$$f(1) = \frac{1+5}{(1)^2 + (1) - 6} = \frac{6}{-4} = \frac{-3}{2}$$

Pick points right of x = 2:

$$f(3) = \frac{3+5}{(3)^2 + (3) - 6} = \frac{8}{6} = \frac{4}{3}$$

$$f(5) = \frac{5+5}{(5)^2+(5)-6} = \frac{10}{24} = \frac{5}{12}$$



4. Connect the dots

(keeping the asymptotes in mind)

Vertical Asymptotes <u>can not</u> be crossed ever

Horizontal Asymptotes <u>can</u> be crossed EXCEPT at the ends